Passivity-based Pose Synchronization Using Only Relative Pose Information under General Digraphs

Tatsuya Ibuki, Takeshi Hatanaka and Masayuki Fujita

Abstract—This paper investigates pose synchronization on the Special Euclidean group $SE(3)$. We first introduce a passivity-based distributed velocity input law to achieve pose synchronization presented in our previous works, where the velocity input partially makes use of absolute pose information to cancel the coupling term between position and orientation. In this paper, we present a new velocity input based only on relative pose information and conduct convergence analysis based on stability of perturbed systems. Moreover, we show a necessary and sufficient condition in terms of interconnection topologies for pose synchronization on $SE(3)$. We finally demonstrate the effectiveness of the present velocity input through simulations.

I. INTRODUCTION

The Great East Japan Earthquake wreaked severe damage to Japan on March 11th, 2011 [1]. Especially, radioactive contamination caused by the damage of nuclear power plants, such as air pollution, threatens health of organism. Measurement of pollution levels is thus an urgent need in Japan for the future (see e.g. [2]). However, it is difficult for existing technologies using static sensors to measure them efficiently since such pollution spreads through a wide area.

A large amount of research works has been devoted to a new technology called mobile sensor network [3]-[5]. Mobile sensor networks are collections of interconnected multiple mobile sensors with computing capability, which thus have potential advantages in performances and robustness against failures especially in dynamical environments. In practice, each mobile sensor is often required to behave cooperatively each other by using only limited information so that the group achieves specified behaviors.

Cooperative control gives fundamentals to meet the above requirement [6]-[9]. Cooperative control problems for mobile sensor networks have been formulated as 2D or 3D pose (position and attitude) coordination problems [3]-[5]. In this paper, we investigate a pose synchronization problem as one of such problems whose objective is to lead agents’ poses to a common value by utilizing distributed control strategies. In the stage of implementation of the strategies, it is desired for each agent to make use of only relative pose information with respect to its neighbors measured by relative sensing devices without communication.

Synchronization problems on $SO(3)$ or $SE(3)$ are investigated in [10]-[18]. [10]-[13] handle multiple rigid bodies with attitude dynamics represented by Euler-Lagrange equations. Whereas, [14] considers general Lagrangian systems.

T. Ibuki, T. Hatanaka and M. Fujita are with the Department of Mechanical and Control Engineering, Tokyo Institute of Technology, Tokyo 152-8550, JAPAN fujita@ctrl.titech.ac.jp

[10] proposes attitude synchronization laws on $SO(3)$ based only on relative attitude information, but it assumes that interconnection topologies among agents are undirected. [12]-[14] present synchronization laws with milder assumptions of topologies, however, the proposed laws consist of information other than relative attitudes. [11] tackles pose synchronization problems on $SE(3)$, but it considers undirected topologies. On the other hand, we have presented passivity-based synchronization laws on $SO(3)$ [16] and $SE(3)$ [17] under milder topology assumptions than [10], [11]. Although the control law in [16] is based only on relative attitude information, [17] partially uses absolute pose information to cancel the coupling term between position and orientation.

Using only relative pose information, we tackle a pose synchronization problem on $SE(3)$ in this paper. We first introduce a passivity-based distributed velocity input law to achieve pose synchronization proposed in our previous work [17]. We then present a new velocity input in the absence of absolute pose information and conduct convergence analysis. Moreover, we show a necessary and sufficient condition in terms of interconnection topologies for pose synchronization. The effectiveness of the present velocity input is demonstrated through simulations. Main contributions of this paper are as follows. (i) We present a fully autonomous velocity input to achieve pose synchronization on $SE(3)$ without using any absolute information under strongly connected digraphs milder than [10], [11]. This helps implementation as in [18], [19]. (ii) The result is extended to a much wider class of digraphs. (iii) We also prove that the graph condition is necessary and sufficient for synchronization as long as the graph is assumed to be fixed.

II. PRELIMINARIES

A. Interconnection Topologies

In this paper, we consider a network of $n$ rigid bodies. The interconnection topology of networked rigid bodies is represented by a weighted digraph $G = (V, E, W)$, where $V := \{1, \ldots, n\}$ is the node set, $E \subset V \times V$ is the edge set containing pairs of nodes that represent information flow and $W$ is the weight set. The neighbors of body $i$ are defined as $N_i := \{j \in V \mid (j, i) \in E\}$. $N_i$ means that body $i$ receives information from body $j$ if $j \in N_i$. The weights $w_{ij} (> 0) \in W$ represent the reliability or importance of each sensed relative information. We moreover define the weighted graph Laplacian matrix of digraph $G$ as $L_w$ (refer to [17]).

We next define the condensation digraph of $G$, denoted by $G'$, based on [6], [7]. We first define subgraphs $G'_i \subseteq G$ as strongly connected components of $G$ if $G'_i$ is strongly
connected and any other subgraph of $G$ strictly containing $G_i$ is not strongly connected, and $G_i := \{(j), 0, 0\}$ if node $j$ of $G$ does not form any strongly connected component. We next consider each $G_i$ as node $H_i$ of $G'$ and define $(H_i, H_j)$ as edges of $G'$ if there exists at least one directed edge in $G$ from a node of $G_i$ to a node of $G_j$. Then, condensation digraph $G'$ has the following properties.

- Condensation digraph $G'$ is acyclic.
- If digraph $G$ contains a directed spanning tree, then $G'$ contains a directed spanning tree.
- If $G'$ contains a directed spanning tree, then the root of the tree, denoted by $H_i$, is uniquely determined.
- If $G'$ contains a directed spanning tree, then there exists at least one node $H_i$ in $G'$ having the root $H_1$ of the tree as only one parent node.

These properties can be confirmed by the definition of $G'$. Fig. 1 illustrates the definition and properties.

B. Stability of Perturbed Systems

We introduce a part of stability theory of perturbed systems in Section 9.3 of [20] necessary for deriving our main results.

Consider the system

$$\dot{x} = f(t, x) + g(t, x),$$

where $f : [0, \infty) \times D \rightarrow \mathbb{R}^n$ and $g : [0, \infty) \times D \rightarrow \mathbb{R}^n$ are piecewise continuous in time $t$ and locally Lipschitz in $x$ on $[0, \infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. We regard this system as a perturbation of the following nominal system.

$$\dot{x} = f(t, x).$$

We also suppose the perturbation term $g(t, x)$ satisfies the following bound.

$$\|g(t, x)\| \leq \delta(t), \quad \forall t \geq 0, \quad \forall x \in D.$$ (3)

Here, $\delta : \mathbb{R} \rightarrow \mathbb{R}$ is nonnegative, continuous and bounded for all $t \geq 0$. Then, from Lemma 9.4 and 9.6 in [20], the following proposition holds.

Proposition 1: Let $x = 0$ be an exponentially stable equilibrium point of nominal system (2) with a Lyapunov function $V(t, x)$ such that $c_1 \|x\|^2 \leq V(t, x) \leq c_2 \|x\|^2$, $\dot{V} \leq -c_3 \|x\|^2$ and $\|\partial V/\partial x\| \leq c_4 \|x\|$ for all $(t, x) \in [0, \infty) \times D$ for some positive constants $c_1$, $c_2$, $c_3$ and $c_4$. Here, $D := \{x \in \mathbb{R}^n | \|x\| < r\}$ for a positive constant $r$. Suppose the perturbation term $g(t, x)$ satisfies (3) and $\lim_{t \rightarrow \infty} \delta(t) = 0$ holds. Then, provided $x(t_0)$ satisfies $\|x(t_0)\| \leq r \sqrt{c_1/c_2}$, the solution of perturbed system (1) satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

III. POSE SYNCHRONIZATION

A. Rigid Body Motion and Control Objective

Throughout this paper, we consider a network of $n$ rigid bodies in three-dimensional space (see Fig. 2). Let $\Sigma_w$ be an inertial coordinate frame and $\Sigma_i$, $i \in \mathcal{V}$ body-fixed frames. We denote the pose of body $i$ in $\Sigma_w$ by $(p_{wi}, e^{\xi_{wi}}\theta_{wi}) \in SE(3)$ or homogeneous representation

$$g_{wi} = \begin{bmatrix} e^{\xi_{wi}\theta_{wi}} & p_{wi} \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 4}, \quad i \in \mathcal{V}.$$ 

Here, $\xi_{wi} \in \mathbb{R}^3$ (\(|\xi_{wi}\| = 1\)) and $\theta_{wi} \in \mathbb{R}$ specify the rotation axis and angle, respectively. The notation ‘
\' denotes the inverse operator to ‘\’. For simplicity, we use $\hat{\xi} \theta_{wi}$ to denote $\hat{\xi}_{wi} \theta_{wi}$. We next introduce the velocity of each rigid body to represent rigid body motion of $\Sigma_i$ relative to $\Sigma_w$. We define the body velocity of body $i$ relative to $\Sigma_w$ as $V_{wi} = (\dot{p}_{wi}, \dot{\omega}_{wi}) := (g_{wi}^{-1} \dot{g}_{wi})' \in \mathbb{R}^6$, or

$$\dot{V}_{wi} := g_{wi}^{-1} \dot{g}_{wi} = \begin{bmatrix} \dot{\omega}_{wi} & \dot{p}_{wi} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4}, \quad i \in \mathcal{V},$$

where $\dot{p}_{wi} \in \mathbb{R}^3$ and $\dot{\omega}_{wi} \in \mathbb{R}^3$ represent the linear and angular velocities, respectively. Then, rigid body motion is represented by the following kinematic model.

$$\dot{g}_{wi} = g_{wi} \dot{V}_{wi}, \quad i \in \mathcal{V}.$$ (4)

We also define the pose and body velocity of $\Sigma_j$ relative to $\Sigma_i$ as $g_{ij} = (p_{ij}, e^{\phi_{ij}}) := g_{wi}^{-1} g_{wj} \in SE(3)$, and $V_{ij} := (\dot{g}_{ij}^{-1} \dot{g}_{ij})' \in \mathbb{R}^6$, respectively.

We next define pose synchronization as follows.

1\ We can use $\xi_{wi} := p_{wi} + d_i$, instead of $p_{wi}$ as a virtual position, where $d_i \in \mathbb{R}^3$ is a position bias of rigid body $i$ as in [17]. Then, the final position configuration after synchronization is formed by $d_i - d_j$. [4710]
Definition 1: A group of n rigid bodies is said to achieve pose synchronization if
\[
\lim_{t \to \infty} \psi(g_{ij}) = 0 \quad \forall i, j \in \mathcal{V},
\]
where
\[
\psi(g_{ij}) := (1/2)\|p_{ij}\|^2 + \phi(e^{\xi_{ij}}) \geq 0, \quad \phi(e^{\xi_{ij}}) := (1/2)(I_d - e^{\xi_{ij}}) \geq 0 \quad \text{is the pose error energy,}
\]
and \(V_d = (v_d, \omega_d) \in \mathbb{R}^6\) where the vectors \(v_d \in \mathbb{R}^3\) and \(\omega_d := (e^{\xi_d}e^{-\xi_d})' \in \mathbb{R}^3\) respectively represent the desired linear and angular velocities. From the definition of function \(\psi\), equation (5) implies that poses of all bodies eventually converge to a common value (Fig. 3) and all bodies move with \((v_d, \omega_d)\). Then, the objective of this paper is to propose a velocity input to achieve (5) based only on relative pose information \(g_{ij}\).

**B. Review of Pose Synchronization**

In order to achieve pose synchronization, the following body velocity input is proposed in [17].
\[
V^{b}_{wi} = K_i \sum_{j \in \mathcal{N}_i} \left[ \begin{array}{c} w_{ij} \\ p_{ij} \\ sk(e^{\xi_{ij}})' \end{array} \right] = \left[ e^{-\xi_{wi}v_d}, e^{-\xi_{wi}\omega_d} \right], \quad i \in \mathcal{V},
\]
where \(K_i = \left[ \begin{array}{ccc} k_{pi}I_3 & 0 & 0 \\ k_{ci}I_3 & 0 & 0 \end{array} \right] \), \(k_{pi}, k_{ci} > 0\), and \(sk(e^{\xi_{ij}}) := (1/2)(e^{\xi_{ij}} - e^{-\xi_{ij}})\). Velocity input (6) consists of two parts. The first term assures pose synchronization in the sense of (5). The second one specifies a desired trajectory after synchronization. Thus, \(v_d\) and \(\omega_d\) should be common among all rigid bodies. Then, the following facts hold.

**Fact 1:** [17] Consider n rigid bodies represented by (4) and suppose that velocity input (6) is applied to each body. If there exists \(e^{\xi_{ij}}\) such that \(e^{-\xi_{ij}v_d}, \forall i \in \mathcal{V}\) are positive definite\(^{2}\) at the initial time, then \(e^{-\xi_{ij}v_d}, \forall i \in \mathcal{V}\) remain to be positive definite for all subsequent time. Also, the same statement holds for \(e^{\xi_{ij}}\), \(\forall i, j \in \mathcal{V}\).

**Fact 2:** [17] Consider n rigid bodies represented by (4). Then, under the assumption that there exists \(e^{\xi_{ij}}\) such that \(e^{-\xi_{ij}v_d}, \forall i \in \mathcal{V}\) are positive definite at the initial time and the fixed interconnection digraph \(G\) is strongly connected, velocity input (6) achieves pose synchronization in the sense of (5) with \((e^{-\xi_{wi}v_d}, e^{-\xi_{wi}\omega_d})\) instead of \((v_d, \omega_d)\).

\(^{2}\)Throughout this paper, we refer to a real square matrix \(M\), not necessarily symmetric, as a positive definite matrix iff \(x^TMx > 0\) holds for all nonzero vector \(x\).

**Remark 1:** The statement of Fact 1 in [17] is proved for fixed and strongly connected digraphs. However, it is immediate to show the statement holds for any fixed digraph in the same way as in [17].

**C. Relative Pose Information-based Pose Synchronization**

In Subsection III-B, we introduced velocity input (6) to achieve pose synchronization. Here, the first and second terms embody cohesion and alignment rules, respectively. If we neglect the alignment rule, i.e. \(V_d = 0\), velocity input (6) is based only on relative pose information. However, for alignment \((V_d \neq 0)\), each rigid body has to share a common frame. In this subsection, we thus use the following velocity input instead of (6).
\[
V^{b}_{wi} = K_i \sum_{j \in \mathcal{N}_i} \left[ \begin{array}{c} w_{ij} \\ p_{ij} \\ sk(e^{\xi_{ij}})' \end{array} \right] + v_d, \quad i \in \mathcal{V},
\]
where we suppose that \(v_d\) is bounded. Velocity input (7) without the first position term is the same as in attitude synchronization problems [16]. Therefore, the present input (7) just adds position feedback to that in [16]. Then, the following theorem holds.

**Theorem 1:** Consider n rigid bodies represented by (4). Then, under the assumption that there exists \(e^{\xi_{ij}}\) such that \(e^{-\xi_{ij}v}e^{-\xi_{ij}w}, \forall i \in \mathcal{V}\) are positive definite at the initial time and the interconnection digraph \(G\) is fixed and strongly connected, velocity input (7) achieves pose synchronization in the sense of (5).

**Proof:** We first consider the position and orientation dynamics of rigid body motion (4) with velocity input (7) individually.
\[
p^{w}_{wi} = e^{\xi_{wi}v}v^{b}_{wi} = \sum_{j \in \mathcal{N}_i} k_{pj}w_{ij}(p_{wi} - p_{w_j}) + e^{\xi_{wi}v}v_d,
\]
\[
e^{\xi_{wi}v} = e^{\xi_{wi}v} + e^{\xi_{wi}w} - e^{\xi_{wi}v}w^{b}_{wi} = e^{\xi_{wi}v} + \sum_{j \in \mathcal{N}_i} k_{ci}w_{ij}sk(e^{\xi_{ij}}) + \omega_d.
\]
Since \(e^{\xi_{wi}w}sk(e^{\xi_{wi}})' = sk(e^{\xi_{wi}})'\) holds, the time derivative of \(\phi(e^{\xi_{wi}})\) along the trajectories of (9) yields
\[
\dot{e}^{\xi_{wi}} = \phi(e^{\xi_{wi}})'T e^{\xi_{wi}} - \phi(e^{\xi_{wi}})'T \omega_d,
\]
\[
= (sk(e^{\xi_{wi}})')'T \left( \sum_{j \in \mathcal{N}_i} k_{ci}w_{ij}sk(e^{\xi_{ij}})' + \omega_d \right) = \left( sk(e^{\xi_{wi}}) \right)' \left( \sum_{j \in \mathcal{N}_i} k_{ci}w_{ij}sk(e^{\xi_{ij}})' + e^{-\xi_{wi}\omega_d} \right).
\]
This means that the difference between \(\omega_d\) and \(e^{-\xi_{wi}\omega_d}\) does not influence the trajectory of \(\phi(e^{\xi_{wi}})\). Therefore, we can use the same analysis as in [17] (Fact 1 and 2). Namely, by defining the function \(U_c := \sum_{j \in \mathcal{V}} \gamma_i/K_{ci} \phi(e^{\xi_{wi}})\), we can show that \(U_c\) is non-positive. Here, \(\gamma_i, i \in \mathcal{V}\) are the elements of vector \(\gamma\) satisfying
\[
\gamma^TL_\omega = 0, \quad \gamma^T = [\gamma_1 \cdots \gamma_n], \quad \gamma_i > 0 \quad \forall i \in \mathcal{V}.
\]
Vector \(\gamma\) exists if the interconnection digraph \(G\) is strongly connected. Thus, from LaSalle’s Invariance Principle [20],
we can conclude that orientations of all rigid bodies asymptotically converge to a common value.

We use Proposition 1 for convergence of positions. Consider the evolution of position errors. Then, we get the following equation from (8).

\[ \dot{p}_{wi} - \dot{p}_{wj} = \sum_{l \in \mathcal{N}_i} k_{pi} w_{il} (p_{wl} - p_{wi}) - \sum_{l \in \mathcal{N}_j} k_{pj} w_{jl} (p_{wl} - p_{wj}) + (e^{\tilde{\theta}_{wi}} - e^{\tilde{\theta}_{wj}}) v_d. \]

We define \( p \in \mathcal{R}^{3n(n-1)} \) as the stuck vector of \( p_{wi} - p_{wj} \) \( \forall i, j \) \( i \neq j \). Then, the dynamics of \( p \) is given by

\[ \dot{p} = Ap + B(1 \otimes v_d), \quad 1 := [1 \cdots 1]^T \in \mathcal{R}^{n(n-1)}, \quad (10) \]

where \( A \in \mathcal{R}^{3(n-1) \times 3n(n-1)} \) is the constant matrix consisting of \( k_{pi} \) and \( w_{ij} \), and \( B \in \mathcal{R}^{3(n-1) \times 3(n-1)} \) is the block diagonal matrix whose diagonal elements are \( e^{\tilde{\theta}_{wi}} - e^{\tilde{\theta}_{wj}} \). Then, from Fact 2 with \( v_d = 0 \), we can conclude that when \( B = 0 \), the origin of (10) is exponentially stable.

Since \( \lim_{t \to \infty} \phi(e^{\tilde{\theta}_{ij}}) = 0 \) holds for all \( i, j \in \mathcal{V} \) and \( v_d \) is bounded, \( \lim_{t \to \infty} B(1 \otimes v_d) = 0 \) holds and thus we conclude that \( \lim_{t \to \infty} p = 0 \) from Proposition 1. We finally note that if the first equation in (5) holds, then the second one is automatically achieved since the first term vanishes in (7). This completes the proof.

**Remark 2:** We write down the position part of relative rigid body motion \( \dot{g}_{ij} \) as follows.

\[ \dot{p}_{ij} = v^b_{wi} + \dot{v}^b_{wi} + e^{-\tilde{\theta}_{wi}} e^{\tilde{\theta}_{wij}} v^b_{wj}. \]

From this equation, we can easily see that the current orientation influences relative position dynamics when each rigid body uses common body velocities. Therefore, [17] eliminates this influence by using \( e^{-\tilde{\theta}_{wi}} \) and proves Fact 2 by utilizing non-increasing properties of appropriate potential functions. On the other hand, this paper allows potential functions to increase in transient by regarding position error dynamics (10) as perturbed systems. This issue does not arise when we confine our focus to only \( SO(3) \).

IV. EXTENSION TO GENERAL DIGRAPHs

In order to obtain a necessary and sufficient condition in terms of fixed interconnection topologies for pose synchronization, we use condensation digraph \( G' \) and stability theory of perturbed systems.

We first analyze a property of condensation digraph \( G' \). Suppose that digraph \( G \) has a directed spanning tree. Then, from the properties of \( G', G' \) also has a directed spanning tree whose root is denoted by \( H_1 \), and \( G' \) is acyclic. Let us now define set \( H_k, k \in \{1, 2, \cdots \} \) collecting nodes \( H_i \) whose maximal path lengths from the root (\( H_1 \)) in \( G' \) are equal to \( k \) (see Fig. 4). We also define \( H_0 := \{H_1\} \). Then, from the definitions of \( G' \) and \( H_k \), all nodes \( H_i \) in \( H_k \) have neighbors only in \( \{H_{k-1}, \cdots, H_0\} \). By using this property, we can take an inductive approach to convergence analysis of pose synchronization for general digraphs, i.e. we prove synchronization in \( H_0 \), then \( H_1 \), \( H_2 \) and so on.

We next consider pose synchronization in one strongly connected group \( H_i \in \mathcal{H}_k \) under a limited situation that all strongly connected groups in \( \{H_{k-1}, \cdots, H_0\} \) have already achieved synchronization and \( v_d = 0 \) (see Fig. 5). This result will be used to prove the main result of this paper. Since all rigid bodies in \( \{H_{k-1}, \cdots, H_0\} \) have a common pose under this situation, we denote the pose by \( g_{00} \). Then, we have the following lemma.

**Lemma 1:** Consider multiple rigid bodies in \( H_i \in \mathcal{H}_k \). Then, under the assumptions that \( e^{\tilde{\theta}_{0j}} \) \( \forall i \) are positive definite at the initial time and there exists at least one body having at least one neighbor body in \( \{H_{k-1}, \cdots, H_0\} \) moving with \( (0, \omega_d) \), the equilibrium \( g_{00} = \bar{T}_i \bar{T}_j \) of all networked systems (4), (7) with \( v_d = 0 \) is exponentially stable.

**Proof:** We consider \( m \) rigid bodies in \( H_i \in \mathcal{H}_k \) and denote the bodies by \( \{1, \cdots, m\} \) in this proof. We define a potential function

\[ U := m \left( \frac{\gamma_i}{2k_{pi}} \| p_{wi} - p_{wj} \|^2 + \frac{\gamma_i}{k_{ei}} \phi(e^{\tilde{\theta}_{wi}}) \right). \]

Then, from the appropriate calculation, we get positive scalars \( a \) satisfying

\[ \dot{U} \leq -aU. \]

This inequality means \( g_{00} = I_4, i \in \mathcal{V}_m \) is exponentially achieved.

**Remark 3:** The setting in Lemma 1 can be regarded as a leader-following case. Though [17] shows achievement of exponential pose synchronization, the analysis cannot be applied to the leader-following case. Therefore, we introduce different potential functions from that in [17] and apply brand-new procedures for exponential stability analysis in Lemma 1.
We finally show a necessary and sufficient condition in terms of fixed interconnection topologies for pose synchronization by using Proposition 1 and Lemma 1.

**Theorem 2:** Consider \( n \) rigid bodies represented by (4). Then, velocity input (7) achieves pose synchronization for all initial states such that \( \dot{\xi}_{ij}^0 > 0 \) \( \forall i, j \in V \) iff the fixed interconnection digraph \( G \) has a directed spanning tree.

**Proof:** Necessity: When we consider the case that \( \dot{\xi}_{ij}^0 > 0 \) \( \forall i, j \in V \), i.e., we consider only position synchronization on \( \mathbb{R}^3 \), the problem is the same as traditional consensus problems. Therefore, the necessary condition is simply derived from the results in [8] (Proof by contraposition: If \( G \) does not have a directed spanning tree, some positions do not synchronize for some initial positions).

**Sufficiency:** We give a sufficient condition by using induction, where we consider the condensation digraph \( G' \) and \( H_k, k \in \{0, 1, \cdots \} \). We first consider \( H_0 \), i.e., the strongly connected group forming the root \( H_1 \) of the directed spanning tree in \( G' \) (refer to Fig. 4). Then, from Theorem 1, all rigid bodies in the group achieve pose synchronization in the sense of (5) and all bodies move with \((v_d, \omega_d)\) after sufficient time.

We next consider each strongly connected group in \( H_1 \). In each present group, some rigid bodies have neighbors in \( H_2 \). We choose one body, denoted by \( l \), among the neighbors. Then, we get the following equation for body \( i \) in the group.

\[
\dot{p}_{wi} - \dot{p}_{wl} = (e^{\xi_{wi}^0} - e^{\xi_{wl}^0})v_d + \sum_{j \in N_i} k_{pi}w_{ij}(p_{wj} - p_{wi})
- \sum_{j \in N_i} k_{pi}w_{ij}(p_{wj} - p_{wl})
+ \sum_{j \in N_i \setminus N_i'} k_{pi}w_{ij}(p_{wj} - p_{wi}) - \sum_{j \in N_i} k_{pi}w_{ij}(p_{wj} - p_{wl})
= \sum_{j \in N_i \setminus N_i'} k_{pi}w_{ij}(p_{wj} - p_{wi}) + \sum_{j \in N_i} k_{pi}w_{ij}(p_{wj} - p_{wl})
+ \sum_{j \in N_i} k_{pi}w_{ij}(p_{wj} - p_{wl}) - \sum_{j \in N_i} k_{pi}w_{ij}(p_{wj} - p_{wl})
+ (e^{\xi_{wi}^0} - e^{\xi_{wl}^0})v_d,
\]

(11)

where \( N_i' \) represents the neighbors of body \( i \) in \( \{H_0\} \). We define \( p_l \) as the stack vector of \( p_{wi} - p_{wl} \) for every body \( i \) in the present group. Then, by noting that \( p_{wj} - p_{wl} \) is rewritten by \( p_{wj} - p_{wl} - (p_{wi} - p_{wl}) \), the dynamics of \( p_l \) is given by

\[
\dot{p}_l = A_p p_l + \eta_p,
\]

(12)

where \( A_p \) is the constant matrix consisting of \( k_{pi} \) and \( w_{ij} \), and \( \eta_p \) is the vector consisting of the last three terms in (11).

Similarly, by defining \( R_l \) as the stack matrix of \( e^{\xi_{0i}} \) for every body \( i \) in the group, the dynamics of \( R_l \) is given by

\[
\dot{R}_l = F(R_l) + \eta_e.
\]

(13)

where \( F(\cdot) \) is the matrix function consisting of \( e^{\xi_{0i}}, k_{ei}, w_{ij}, \omega_d, \) and \( \eta_e \) is the matrix consisting of \( k_{ei}, w_{ij}, e^{\theta_{ij}}, j \in N_i' \) and \( k_{ei}, w_{ij}, e^{\theta_{ij}}, j \in N_i \). It should be noted that \( \lim_{t \to -\infty} \eta_e = 0 \) holds since the root group forming \( H_1 \) achieves pose synchronization. Also, \( \lim_{t \to -\infty} \eta_p = 0 \) holds if the root group achieves pose synchronization and the trajectories of \( p_l \) and \( R_l \) in this situation are equivalent to those in the setting of Lemma 1. Thus, from Lemma 1, the equilibria \( p_l = 0 \) and \( R_l = 1 \otimes I_3 \) for systems (12) and (13) are exponentially stable when \( \eta_p = 0 \) and \( \eta_e = 0 \) hold respectively. Namely, if we regard \( \eta_p \) and \( \eta_e \) as perturbations of systems (12) and (13) respectively, then, from Proposition 1, we conclude that all poses in the present group eventually converge to the same pose as the common one in \( H_1 \) and all rigid bodies move with \((v_d, \omega_d)\). This means pose synchronization is achieved among the strongly connected groups in \( \{H_1, H_0\} \).

We next assume that each strongly connected group in \( H_q \) \( (q \geq 2) \) achieves pose synchronization among the groups in \( \{H_q, H_{q-1}, \cdots, H_0\} \) and consider each group in \( H_{q+1} \). Then, from the definition of \( H_{q+1} \), some rigid bodies in the present group definitely have neighbors in \( H_q \) and may have in \( \{H_{q-1}, \cdots, H_0\} \). Thus, by choosing one body among such neighbors, we obtain the same pose errors formulation as (12) and (13). Also, since pose synchronization is achieved among the groups in \( \{H_q, H_{q-1}, \cdots, H_0\} \), \( \eta_p \) and \( \eta_e \) in this case also eventually converge to 0. Namely, by the same analysis as for \( H_1 \), we can show that each strongly connected group in \( H_{q+1} \) also achieves pose synchronization among the groups in \( \{H_{q+1}, \cdots, H_0\} \). This completes the proof.

**Remark 4:** This condition is well known in consensus on a vector space. The contribution of Theorem 2 is to prove that this condition is also necessary and sufficient for synchronization on \( SE(3) \), although most of works for synchronization on \( SO(3) \) or \( SE(3) \) make use of more restrictive graph assumptions, in particular when each rigid body uses only relative pose information [10]-[17].

**V. Verification**

In this section, we demonstrate the effectiveness of the present velocity input (7) through simulations in three-dimensional space.
Consider 20 rigid bodies with the interconnection topology depicted in Fig. 6 which has a directed spanning tree. The velocity input (7) with \( k_{vi} = k_{ei} = 1 \) \( \forall i \in \mathcal{V} \), \( w_{ij} = 1 \) \( \forall (j, i) \in \mathcal{E} \), \( v_{d} = [0 \ 0 \ 1]^T \) [m/s] and \( \omega_{d} = [0.5 \ 0 \ 0]^T \) [rad/s] is applied to each body under random initial poses such that \( e^{\xi \theta_{wi}} > 0 \) \( \forall i, j \in \mathcal{V} \).

Simulation results are shown in Figs. 7-12. Figs. 7-9 and Figs. 10-12 illustrate time responses of positions and orientations in \( \Sigma_w \), respectively. From these figures, we can see that poses of all rigid bodies asymptotically converge to a common value and all bodies move with \( (v_{d}, \omega_{d}) \) after synchronization. Namely, pose synchronization is achieved by velocity input (7). We can also see that pose synchronization is achieved in first \( \mathbf{H}_0 \), then \( \mathbf{H}_1, \mathbf{H}_2 \), and so on.

**VI. CONCLUSIONS**

In this paper, we have investigated pose synchronization on the Special Euclidean group \( SE(3) \). We have first introduced a passivity-based distributed velocity input to achieve pose synchronization proposed in our previous work [17]. We have then presented a new velocity input based only on relative pose information and conducted convergence analysis in the absence of absolute pose information. Moreover, we have shown a necessary and sufficient condition in terms of fixed interconnection topologies for pose synchronization on \( SE(3) \). We have finally demonstrated the effectiveness of the present velocity input through simulations.

Further directions of this research are as follows as in [17]. (i) We present relative pose information-based flocking algorithms embodying collision avoidance (separation rule for flocking) and analyze the performance. (ii) We conduct convergence analysis of pose synchronization under switching interconnection topologies.

**REFERENCES**